3
Simple Random Sampling

3.1 INTRODUCTION

Everyone mentions simple random sampling, but few use this method for population-based surveys. Rapid surveys are no exception, since they too use a more complex sampling scheme. So why should we be concerned with simple random sampling? The main reason is to learn the theory of sampling. Simple random sampling is the basic selection process of sampling and is easiest to understand.

If everyone in a population could be included in a survey, the analysis featured in this book would be very simple. The average value for equal interval and binomial variables, respectively, could easily be derived using Formulas 2.1 and 2.3 in Chapter 2. Instead of estimating the two forms of average values in the population, they would be measuring directly. Of course, when measuring everyone in a population, the true value is known; thus there is no need for confidence intervals. After all the purpose of the confidence interval is to tell how certain the author is that a presented interval brackets the true value in the population. With everyone measured, the true value would be known, unless of course there were measurement or calculation errors.

When the true value in a population is estimated with a sample of persons, things get more complicated. Rather then just the mean or proportion, we need to derive the standard error for the variable of interest, used to construct a confidence interval. This chapter will focus on simple random sampling or persons or households, done both with and without replacement, and present how to derive the standard error for equal interval variables, binomial variables, and ratios of two variables. The latter, as described earlier, is commonly used in rapid surveys and is termed a *ratio estimator*. What appears to be a proportion, may actually be a ratio estimator, with its own formula for the mean and standard error.

3.1.1 Random sampling

Subjects in the population are sampled by a random process, using either a random number generator or a random number table, so that each person remaining in the population has the same probability of being selected for the sample. The process for selecting a random sample is shown in Figure 3-1.

Figure 3-1

The population to be sampled is comprised of nine units, listed in consecutive order from one to nine. The intent is to randomly sample three of the nine units. To do so, three random numbers need to be selected from a random number table, as found in most statistics texts and presented in Figure 3-2. The random number table consists of six columns of two-digit non-repeatable numbers listed in random order. The intent is to sample three numbers between 1 and 9, the total number in the population. Starting at the top of column A and reading down, two numbers are selected, 2 and 5. In column B there are no numbers between 1 and 9. In column C the first random number in the appropriate interval is 8. Thus in our example, the randomly selected numbers are 2, 5 and 8 used to randomly sample the subjects in Figure 3-1. Since the random numbers are mutually exclusive (i.e., there are no duplicates), each person with the illustrated method is only sampled once. As described later in this chapter, such selection is sampling *without replacement*.

Figure 3-2

Random sampling assumes that the units to be sampled are included in a list, also termed a sampling *frame*. This list should be numbered in sequential order from one to the total number of units in the population. Because it may be time-consuming and very expensive to make a list of the population, rapid surveys feature a more complex sampling strategy that does not require a complete listing. Details of this more complex strategy are presented in Chapters 4 and 5. Here, however, every member of the population to be sampled is listed.

3.1.2 Nine drug addicts

A population of nine drug addicts is featured to explain the concepts of simple random sampling. All nine addicts have injected heroin into their veins many times during the past weeks, and have often shared needles and injection equipment with colleagues. Three of the nine addicts are now infected with the human immunodeficiency virus (HIV). To be derived are the proportion who are HIV infected (a binomial variable), the mean number of intravenous injections (IV) and shared IV injections during the past two weeks (both equal interval variables), and the proportion of total IV injections that were shared with other addicts. This latter proportion is a ratio of two variables and, as you will learn, is termed a *ratio estimator*.

Figure 3-3

The total population of nine drug addicts is seen in Figure 3-3. Names of the nine male addicts are listed below each figure. The three who are infected with HIV are shown as cross-hatched figures. Each has intravenously injected a narcotic drug eight or more times during the past two weeks. The number of injections is shown in the white box at the midpoint of each addict. With one exception, some of the intravenous injections were shared with other addicts; the exact number is shown in Figure 3-3 as a white number in a black circle.

Our intention is to sample three addicts from the population of nine, assuming that the entire population cannot be studied. To provide an unbiased view of the population, the sample mean should on average equal the population mean, and the sample variance should on average equal the population variance, corrected for the number of people in the sample. When this occurs, we can use various statistical measures to comment about the truthfulness of the sample findings. To illustrate this process, we start with the end objective, namely the assessment of the population mean and variance.

Population Mean. For total intravenous drug injections, the mean in the population is derived using Formula 3.1

$$\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N} \tag{3.1}$$

where X_i is the total injections for each of the *i* addicts in the population and *N* is the total number of addicts. Thus, the mean number of intravenous drug injections in the population shown in Figure 3-3 is

$$\bar{X} = \frac{10 + 8 + 12 + 9 + 11 + 11 + 9 + 11 + 10}{9} = \frac{91}{9} = 10.1$$

or 10.1 intravenous drug injections per addict.

Population Variance. Formula 3.2 is used to calculate the variance for the number of intravenous drug injections in the population of nine drug addicts.

$$\sigma^2 = \frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N} \tag{3.2}$$

where σ^2 is the Greek symbol for the population variance, X_i and *N* are as defined in Formula 3.1 and $\bar{X}$ is the mean number of intravenous drug injections per addict in the population. Using Formula 3.2, the variance in the population is

$$\sigma^2 = \frac{(10 - 10.1)^2 + (8 - 10.1)^2 .... (11 - 10.1)^2 + (10 - 10.1)^2}{9} = 1.43$$

Sample Mean. Since the intent is to make a statement about the total population of nine addicts, a sample of three addicts will be drawn, and their measurements will be used to represent the group.

The three will be selected by simple random sampling. The mean for a sample is derived using Formula 3.4.

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{3.4}$$

where x_i is the number of intravenous injections in each sampled person and *n* is the number of sampled persons. For example, assume that Roy-Jon-Ben is the sample. Roy had 12 intravenous drug injections during the past two weeks (see Figure 3-3), Jon had 9 injections and Ben had 10 injections. Using Formula 3.4,

$$\bar{x} = \frac{12 + 9 + 10}{3} = 10.3$$

the sample estimate of the mean number of injections in the population (seen previously as 10.1) is 10.3.

Sample Variance. The variance of the sample is used to estimate the variance in the population and for statistical tests. Formula 3.5 is the standard variance formula for a sample.

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \tag{3.5}$$

where s^2 is the symbol for the sample variance, x_i is the number of intravenous injections for each of the *i* addicts in the sample and $\bar{x}$ is the mean intravenous drug injections during the prior week in the sample. For the sample Roy-Jon-Ben with a mean of 10.3, the variance is

$$s^2 = \frac{(12 - 10.3)^2 + (9 - 10.3)^2 + (10 - 10.3)^2}{3-1} = 2.33$$

3.2 WITH OR WITHOUT REPLACEMENT

There are two ways to draw a sample, with or without replacement. *With replacement* means that once a person is selection to be in a sample, that person is placed back in the population to possibly be sampled again. *Without replacement* means that once an individual is sampled, that person is not placed back in the population for re-sampling. An example of these procedures is shown in Figure 3-4 for the selection of three addicts from a population of nine. Since there are three persons in the sample, the selection procedure has three steps. Step one is the selection of the first sampled subject, step two is the selection of the second sampled subject and step three is the selection of the third sampled subject. In sampling *with replacement* (Figure 3-4, top), all nine addicts have the same probability of being selected (i.e., 1 in 9) at steps one, two and three, since the selected addict is placed back into the population before each step. With this form of sampling, the same person could be sampled multiple times. In the extreme, the sample of three addicts could be one person selected three times.

Figure 3-4

In sampling *without replacement* (WOR) the selection process is the same as at step one – that is each addict in the population has the same probability of being selected (Figure 3-4, bottom). At step two, however, the situation changes. Once the first addict is chosen, he is not placed back in the population. Thus at step two, the second addict to be sampled comes from the remaining eight addicts in the population, all of whom have the same probability of being selected (i.e., 1 in 8). At the third step, the selection is derived from a population of seven addicts, with each addict having a probability of 1 in 7 of being selected. Once the steps are completed, the sample contains three different addicts. Unfortunately, the reduced selection probability from the first to the third step is at odds with statistical theory for deriving the variance of the sample mean. Such theory assumes the sample was selected *with replacement*. Yet in practice, most simple random samples are drawn *without replacement*, since we want to avoid the strange assumption of one person being tallied as two or more. To resolve this disparity between statistical theory and practice, the variance formulas used in simple random sampling are changed somewhat, as described next.

3.2.1 Possible samples With Replacement.

When drawing a sample from a population, there are many different combinations of people that could be selected. Formula 3.6 is used to derive the number of possible samples drawn *with replacement*,

$$N^n \tag{3.6}$$

where *N* is the number in the total population and *n* is the number of units being sampled. For example when selecting three persons from the population of nine addicts shown in Figure 3-3, the sample could have been Joe-Jon-Hall, or Sam-Bob-Nat, or Roy-Sam-Ben, or any of many other combinations. To be exact, in sampling *with replacement* from the population shown in Figure 3.3, there are

$$N^n = 9^3 = 729$$

or 729 different combinations of three addicts that could have been selected.

Figure 3-5

The frequency distribution of the mean number of IV drug injections of the 729 possible samples selected *with replacement* is shown in the top section of Figure 3-5. Notice that the distribution has a bell shape, similar to a normal curve. There are three notable features of these 729 possible samples.

Notable feature one. While the range of the 729 possible sample means is from a low of 8 to a high of 12, the average value of the sample means for the intravenous drug injections during the prior week is 10.1, the same as the population mean calculated previously with Formula 3.1. That is, when sampled *with replacement*, on average the sample mean provides an unbiased estimate of the population mean.

Notable feature two. The average variance of the 729 possible samples of three selected *with replacement* is equal to the population variance of the nine drug addicts (see Formula 3.2), as shown in Formula 3.7

$$\sigma^2 = \frac{\sum_{i=1}^{729} s_i^2}{729} = 1.43 \qquad (3.7)$$

where s_i^2 is the variance of sample i, where i goes from 1 to 729, the total number of possible samples when selecting three from nine *with replacement*.

Notable feature three. For random samples of size *n* selected from an underlying population *with replacement*, the variance of the mean of all possible samples is equal to the variance of the underlying population divided by the sample size. For the 729 possible samples, the average variance of the mean for a sample of three from an underlying population of nine is shown in Formula 3.8.

$$v(\bar{x}) = \frac{\sum_{i=1}^{729} (\bar{x}_i - \bar{X})^2}{729} = \frac{\sigma^2}{n} = \frac{1.43}{3} = 0.48 \qquad (3.8)$$

Thus with this form of sampling, on average the variance of the sample mean provides an unbiased estimate of the variance of the population divided by the sample size.

Given these three features – namely that the mean, sample variance, and variance of the sample mean are unbiased estimators of the mean, population variance, and variance of the population divided by the sample size – it would seem that sampling *with replacement* is very useful. But is such sampling usually done?

Without Replacement. In the realistic world of sampling, subjects are typically not included in the sample more than once. Also, the order in which subjects are selected for a survey is not important (that is, Roy-Sam-Ben is considered the same as Sam-Ben-Roy). All that matters is if the subject is in or out of the sample. Hence in most surveys, samples are selected *disregarding order* and *without replacement*. But does sampling *without replacement* provide unbiased estimators of the population mean and variance? The answer is "yes," but needing some additional modifications, to be presented next.

Formula 3.9 is used to calculated the number of possible samples that can be drawn *without replacement, disregarding order*,

$$\frac{N!}{n!\ (N-n)!} \tag{3.9}$$

where *N* is the number of people in the population, *n* is the number of sampled persons, and ! is the factorial notation for the sequential multiplication of a number times a number minus 1, continuing until reaching 1. That is, *N!* (termed "*N* factorial") is *N* times *N-1* times *N-2* and the like with the last number being 1.

In our example, we are selecting *without replacement* and *disregarding order* a sample of three addicts from a population of nine addicts (see Figure 3-3). Using Formula 3.9, we find there are

$$\frac{9!}{3!\,(9-3)!} = \frac{9\,x\,8\,x\,7\,x\,6!}{(3\,x\,2\,x\,1)\,x\,6!} = \frac{504\,x\,6!}{6\,x\,6!} = \frac{504}{6} = 84$$

or 84 possible samples. Fortunately when using Formula 3.9, all factorial numbers do not have to be multiplied. For example, the *9!* in the numerator can be converted to *9 x 8 x 7 x 6!,* and the *3! x (9-3)!* in the denominator can be converted to 3 x *2 x 1 x 6!*. By dividing *6!* in the numerator by *6!* in the denominator to get 1, the formula is reduced to *9 x 8 x 7* divided by 3 x *2 x 1* or 84 possible samples.

The distribution of all possible sample means for the 84 samples selected *with replacement, disregarding order* in shown in the bottom section of Figure 3-5, below the distribution of the 729 possible sample means selected *with replacement.* Are the two distributions similar? It is hard to tell since the scale does not permit an easy visual comparison. Figure 3-6 shows the same two distributions, but as a percentage of the total number of possible samples (i.e., 729 *with replacement* and 84 *without replacement*).

Figure 3-6

There are two things to notice. First, the mean of all possible samples selected *with replacement* (i.e., 10.1) is equal to the mean of all samples selected *without replacement*, and both sample means are equal to the population mean. Thus, the sample mean on average remains an unbiased estimator of the population mean when sampling *without replacement*. Second, the percentage distributions of those selected with and without replacement are similar in shape, but there are fewer outlying samples among those sampled *without replacement*. That is, there is less variability among the 84 possible samples selected *without replacement* than the 729 possible samples selected *with replacement*. The reduced variability in sampling *without replacement* is addressed in two ways, namely with a change in the variance formula for the population variance and in the addition of a finite population correction factor (FPC).

First, different from Formula 3.2, the population variance that is being estimated by the sample variance when sampling *without replacement* has a different denominator (N-1), as shown in Formula 3.10.

$$S^2 = \frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N-1} \tag{3.10}$$

where S^2 is the modified population variance and X_i, N and $\bar{X}$ are as defined previously. For the population of nine drug addicts, the modified variance is

$$S^2 = \frac{(10-10.1)^2 + (8-10.1)^2 \ldots\ldots (11-10.1)^2 + (10-10.1)^2}{8} = 1.61$$

When sampling *without replacement* the average variance of all 84 possible samples is equal to the modified population variance (see Formula 3.11).

$$\frac{\sum_{i=1}^{84} s_i^2}{84} = 1.61 = S^2 \tag{3.11}$$

where s_i^2 is the variance in sample *i*, with *i* going from 1 to 84, the total number of possible samples when selecting three from nine *without replacement*.

Second, the variance of the sample mean of all 84 possible samples when sampling *without replacement* is equal to the modified population variance divided by the sample size (as mentioned in ***notable feature three*** in sampling *with replacement*) times a correction factor that accounts for the shrinkage in variance. This correction factor, termed the *finite population correction* (FPC) is shown in Formula 3.12.

$$FPC = \frac{N - n}{N} = 1 - \frac{n}{N} \tag{3.12}$$

where *N* is the size of the population and *n* is the size of the sample. In samples where the sample size is large in relation to the population (an example being a sample of three from a population of nine), the FPC reflects the reduction in variance that occurs when sampling *without replacement* (i.e., with 84 possible samples in the example) compared to sampling *with replacement* (i.e., with 729 possible samples in the example). This reduction in variability when sampling *without replacement* was observed in Figure 3.6, and in the comment that there were fewer outliers in the *without replacement* group.

For the 84 possible samples, the average variance of the mean for a sample of three from an underlying population of nine is shown in Formula 3.13.

$$v(\bar{x}) = \frac{\sum_{i=1}^{84} (\bar{x}_i - \bar{X})^2}{84} = \frac{S^2}{n}\left(1 - \frac{n}{N}\right) = \frac{1.61}{3}\left(1 - \frac{3}{9}\right) = 0.36 \tag{3.13}$$

Notice that *n/N* is the fraction of the population that is sampled. Therefore the *FPC* is often described by sampling specialists as "one minus the sampling fraction." Notice also that the variance of the average samples mean is 0.36 for sampling *without replacement* compared to 0.48 (see Formula 3.8) when sampling *with replacement*, resulting in smaller estimates of sampling error and greater efficiency in the sampling process when the sampling fraction is large. Finally, note that if the sampling fraction is very small, as occurs in typical rapid surveys of few persons drawn from a large population, then the finite population FPC term reduces to approximately 1, and is no longer needed.

In summary, when sampling *without replacement* (i.e., the more practical and typical form of sampling) there are also three notable features, related but not entirely the same as stated earlier in the section on sampling *with replacement.*

Notable feature one. When sampled *without replacement*, on average the sample mean provides an unbiased estimate of the population mean. This feature is the same whether sampling with or without replacement.

Notable feature two. The average variance of all possible samples selected *without replacement* is equal to the modified population variance (i.e., N -1 rather than N in the denominator as when sampling with replacement – see Formula 3.2 versus 3.10).

Notable feature three. For random samples of size *n* selected *without replacement* from an underlying population, the variance of the mean of all possible samples is equal to the modified variance of the underlying population divided by the sample size, multiplied by the finite population correction (FPC) factor.

These three features account for the ability on average of samples selected *with replacement* to truthfully describe an underlying population, and to provide statistical measures of random error in the sampling process.

In conclusion, what has been presented so far is that when drawing a simple random sample from a population, the selected sample is only one among many possible samples. Yet if the sample is selected in an unbiased manner, the average value of all possible samples is the same as the true value in the population. Since the true value is not know and only one sample is being selecting, the variability in the sampling process needs to be described, providing a measure of possible random error. Finally, when sampling *without replacement* the variability of all possible sample means is less than the variability of the sample means when selecting samples *with replacement*, especially when the sampling fraction is large. This reduction in variance is accounted for by the FPC term and results in greater efficiency in the sampling process, but only when the sampling fraction is large. As mentioned in Chapter 2, we will be using formulas that describe the variability of all possible samples to derive a confidence interval for the sample mean or proportion.

In the following sections we will continue to sample three addicts, again drawn *without replacement* from a population of nine addicts. This time, however, a more extensive set of formulas will be used to calculate the mean and variance of two equal interval variables, a binomial variable and a ratio estimator.

3.3 AVERAGE VALUE AND STANDARD ERROR

Every population to be sampled has a true value for the variable of interest. A sample is drawn from the population to estimate this true value. This sample could be viewed as a selection of units from a population of units. Or it could be viewed as the selection of one sample from a population of *all possible samples*. In this section, we will determine the distribution of all possible samples for four variables: total injections, shared injections, HIV infection, and the ratio of shared to total injections. We will derive the mean, standard error and confidence interval for all possible samples of three addicts sampled without replacement from the nine addicts (see Figure 3-3). Since the sample is drawn *without replacement*, there are 84 possible samples.

3.3.1 Equal Interval Data

Each addict in the population of nine injected himself with drugs multiple times during the past two weeks. Some of the injections were shared with other addicts. The total number of injections and the number of shared injections are both equal interval variables, as described in Chapter 2. Different from binomial variables, equal intervals variables have many outcomes ranging in equal intervals from 0 to the upper end of a scale.

Total Injections. The first of the two equal interval variables to be analyzed is *total intravenous drug injections*. The data are shown in white squares for each addict in Figure 3-3. As noted using Formula 3.1, the mean number of intravenous injections per addict in the population of nine drug addicts is

$$\bar{X} = \frac{10+8+12+9+11+11+9+11+10}{9} = \frac{91}{9} = 10.1$$

or 10.1 injections per addict. The distribution of the total injections in the population of nine addicts is shown in Figure 3-7.

Figure 3-7

With the small population of nine, there are 84 possible samples of three addicts that could be selected, assuming sampling *without replacement* and disregarding order. To be derived for each possible sample are the mean for total injections (termed variable x), the standard error of the mean and the confidence interval for total injections. The mean of each of the 84 possible samples is calculated with Formula 3.4. The one sample of Joe-Hal-Roy serves as an example.

$$\bar{x} = \frac{10+8+12}{3} = \frac{30}{3} = 10.0 \textit{ injections per addict}$$

Figure 3-8

The distribution of the 84 possible sample means is shown in the upper left of Figure 3-8. The average value of the 84 possible sample means is 10.1, the same as the mean for the total population of nine addicts (see Figure 3-7). Thus, the sampling scheme for total injections is considered *unbiased*. Observe, however, that some of the possible sample means had values as low as 9 injections while others had values as high as 11.5 injections. By chance alone we could have selected one of these outlying samples, even though the sampling scheme is unbiased.

The top right of Figure 3-8 shows the distribution of the standard errors of 84 sample means. To calculate each standard error, the variance of the sample mean is derived using Formula 3.14.

$$v(\bar{x}) = \frac{N-n}{N} \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n(n-1)} \qquad (3.14)$$

As noted earlier, the term *(N-n)/N* in Formula 3.14 is the *finite population correction*, included only because this is a sample selected *without replacement*. For the sample Joe-Hal-Roy, the variance of the sample mean is

$$v(\bar{x}) = \frac{9-3}{9} \frac{(10-10.0)^2 + (8-10.0)^2 + (12-10.0)^2}{3(3-1)} = 0.89$$

The standard error of the total injections, $se(\bar{x})$, is the square root of the variance, $v(\bar{x})$, as shown in Formula 3.15.

$$se(\bar{x}) = \sqrt{v(\bar{x})} \qquad (3.15)$$

For the sample Joe-Hal-Roy, the standard error is

$$se(\bar{x}) = \sqrt{0.89} = 0.90$$

A confidence interval can be created for each of the 84 possible samples using Formula 3.16.

$$CI(\bar{x}) = \bar{x} \pm t\ (se(\bar{x})) \qquad (3.16)$$

where$\bar{x}$ is as defined previously and *t* is the value of *Student's t* that corresponds to a specified level of confidence (usually 95%) and a sample size of three. But why use a *t-value* when confidence intervals are typically derived with *z-values*, also termed the "standard normal deviate?"

Figure 3-9

The *t-value* is appropriate here because of the exceptionally small size of the sample. As observed in Figure 3-9, for a normal-sized simple random sample of 200 or more, the *t-value* is identical to the *z-value*. For a 95% confidence interval, the *z-value* is 1.96. Yet with a small sample of three, the *t-value* for a 95% confidence interval is 4.30, far greater than 1.96. Using Formula 3.16, the 95% confidence interval for the mean of total injections of 10.0 estimated in the sample Joe- Hal-Roy is

$$CI(\bar{x}) = 10.0 \pm 4.30\ (0.90)$$

or a lower limit of 5.9 and an upper limit of 14.1.

The 84 confidence intervals for all possible samples are shown in the bottom section of Figure 3-8. Five of the 84 possible samples do not bracket the true value (marked with an asterisk at the bottom of the figure) , while 79 (i.e., 94%) do bracket the true value. Keep in mind that only one of the 84 possible samples is selected for the survey. Seventy-nine times out of 84, the constructed confidence interval will contain the true value. Thus in advance of sampling, we can be 94% confident that the interval we construct would contain the true value, assuming the sample was selected in an unbiased manner. If the sample and population had both been much larger, a 95% confidence interval using the *z-value* of 1.96 would have been constructed rather than the 94% confidence interval for 84 possible samples of three with the *t-value* of 4.30.

Shared Injections. The second of the two equal interval variables is *shared intravenous drug injections*. Here the letter *y* is used to describe the statistics rather than *x* as with *total injections*.

Both of these variables will be used in a subsequent section for calculating a ratio estimator. Thus they are identified with different letters, even though the mathematical calculations are identical.

Figure 3-10

The data for shared injections are shown in black circles in Figure 3-3, while the distribution of the shared injections in the population of nine addicts is shown in Figure 3-10.

The population mean for the shared intravenous injections is calculated using Formula 3.17.

$$\bar{Y} = \frac{\sum_{i=1}^{N} Y_i}{N} \tag{3.17}$$

Here $\bar{Y}$ is the mean value of variable Y in the population, Y_i is the value of variable Y in person i, N is the number of persons in the population . The mean number of shared intravenous injections per addict in the population of nine drug addicts is

$$\bar{Y} = \frac{6+2+5+3+0+2+2+5+2}{9} = \frac{27}{9} = 3.0$$

As before, there are 84 possible samples. For each is derived the mean, standard error of the mean and 95% confidence interval. The mean number of shared injections for each sample of three is calculated with Formula 3.18

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \tag{3.18}$$

where $\bar{y}$ is the mean number of intravenous injections per addict in the sample, y_i is the number of shared injections per sampled addict, and n is the number of sampled addicts. Continuing with the example of Joe- Hal- Roy, the mean number of shared injections is

$$\bar{y} = \frac{6+2+5}{3} = \frac{13}{3} = 4.33 \textit{ shared injections per addict}$$

Figure 3-11

The distribution of the 84 possible sample means is shown in the upper left of Figure 3-11. The average value of the 84 possible sample means is 3.0, identical to the mean for the total population of nine addicts. Thus, the sampling scheme for shared injections is *unbiased.* Of course, most of the 84 possible samples have mean values different from 3.0, although the average of all possible samples is the same as the true value of 3.0.

The distribution of the standard errors of the 84 sample means is observed in the upper right of Figure 3-11. The standard errors are derived by first calculating the variance of the sample mean, $v(\bar{y})$, using Formula 3.19.

$$v(\bar{y}) = \frac{N-n}{N} \; \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n(n-1)} \tag{3.19}$$

The *finite population correction, (N-n)/N,* reduces the size of the variance when sampling *without replacement.* For the sample of Joe- Hal-Roy, the variance of the sample mean for shared injections is.

$$v(\bar{y}) = \frac{9-3}{9} \; \frac{(6-4.33)^2 + (2-4.33)^2 + (5-4.33)^2}{3(3-1)} = 0.96$$

The standard error of the total injections, $se(\bar{y})$, is the square root of the variance, $v(\bar{y})$. For the sample Joe-Hal-Roy, the standard error is the square root of 0.96 or 0.98.

The confidence interval for the sample Joe- Hal- Roy for the mean of shared injections of 4.33 is

$$CI(\bar{y}) = 4.33 \pm 4.30\,(0.98)$$

or a lower limit of 0.12 and an upper limit of 8.54. The 84 confidence intervals for all possible samples are shown in the bottom section of Figure 3-11. Five of the confidence intervals do not bracket the true value in the population and are marked with an asterisk. Since 79 (94%) of the 84 possible samples do bracket the true value, the confidence limits in the small example population are actually for 94% confidence intervals. Of course for the one sample that we selected, we would not know if it is among the group of five that do not contain the true interval or 79 that do. After the sample is drawn and an interval is created, the probability is either 0 or 1 that the interval contains the true value. That is, the confidence interval either does or does not bracket the true value. Before the sample is drawn, however, we can be 94% confident that the interval to be calculated will bracket the true value. For larger samples and larger populations, the *z-value* of 1.96 would be used instead to derive intervals which bracket the true value 95 percent of the time.

3.3.2 Binomial Data

The values for binomial data are derived in a similar manner to equal interval data, although the formulas appear slightly different. The binomial variable being considered is HIV infection, a variable with two outcomes, 0 if not infected and 1 if infected. In the population shown in Figure 3-3, three of the nine addicts were infected with HIV. Using Formula 3.1, the mean HIV status in the population of nine drug addicts is

$$P = \frac{1 + 0 + 1 + 0 + 0 + 0 + 0 + 1 + 0}{9} = \frac{3}{9}$$

or 0.33 as presented as a proportion or 33 as a percentage. The mean value of the variable *HIV infection* with an outcome of 0 or 1 is the same as the proportion who are HIV positive. That is, the mean of a binomial variable coded with a 0-1 outcome is the proportion with the attribute. The mean of a binomial variable is also defined as

$$P = \frac{A}{N} \tag{3.20}$$

where *P* is the proportion in the population, *A* is a count of those with the attribute of interest, and *N* is the number of units in the population. In our example, there are three HIV infected persons in a population of nine addicts. Thus the proportion in the population that is infected is 3/9 or 0.33.

Figure 3-12

The distribution of HIV status for the nine addicts is shown in Figure 3-12.

When a binomial variable such as HIV status is coded 0 or 1, Formulas 3.1 and 3.20 are the same. If a binomial variable is coded 1 or 2, however, then the mean value derived with Formula 3.1 will be different than the proportion calculated with Formula 3.20. For the mean of a binomial variable to appear as a proportion (or as a percentage if multiplied by 100), the outcome must be coded as 0 or 1, or tallied as a count of those with the attribute.

Next we calculate the proportion that is infected with HIV for each of the 84 possible samples of three addicts. Formula 3.21 is used to calculate the proportion for each sample.

$$p = \frac{a}{n} \tag{3.21}$$

where *p* is the proportion that is HIV positive in the sample, *a* is the count of HIV infected persons in the sample, and *n* is the number of sampled addicts. Thus for the sample of Joe- Hal-Roy, the proportion that is HIV positive is

$$p = \frac{1+0+1}{3} = \frac{2}{3} = 0.67$$

If by chance this sample would have been selected, the estimate of the proportion who are HIV positive would have been twice the true value of 0.33.

Figure 3-13

The distribution of the 84 possible sample proportions is shown in the upper left of Figure 3-13. The average value of the 84 possible sample proportions is 0.33, the same as the proportion for the total population of nine addicts. When the average value of all possible samples is equal to the proportion (or mean for equal interval variables) in the population, we say that the sampling scheme is *unbiased*. Notice, however, that being unbiased does not imply that the proportion for the one sample being selected is the same as the proportion in the population. Instead being *unbiased* indicates that **on average** the sample proportion equals the population proportion. Also notice that some of the possible sample proportions had values as low as 0 while others had values as high as 1. Thus by chance alone, the sample being selected could have a proportion far different from the true value of 0.33. In fact, only with 45 of the 84 possible samples was the point estimate of the proportion the same as the true value.

Due to the small size of the population and even smaller size of the sample, the proportion in the example does not behave as would occur in a larger population and larger sample. Here 20 of the 84 possible samples have a value of 0 and 1 has a value of 1. The consequence of having a mean value of 0 or 1 will become apparent when calculating the variance, standard error and confidence intervals for the proportion, which follows next.

The variance of a proportion for a simple random sample, selected *without replacement*, is calculated using Formula 3.22,

$$v(p) = \frac{N-n}{N} \; \frac{p\,(1-p)}{n-1} \qquad (3.22)$$

where *v(p)* is the variance of the sample proportion, *p* is the sample proportion, and *n* is the number of sampled addicts. For the sample of Joe- Hal-Roy, the variance of the proportion is

$$v(p) = \frac{9-3}{9} \; \frac{.67\,(1-.67)}{3-1} = 0.074$$

The standard error of the sample proportion, is the square root of the sample variance, as shown in Formula 3.23.

$$se(p) = \sqrt{v(p)} \qquad (3.23)$$

For the sample Joe- Hal-Roy, the standard deviation is the square root of 0.074 or 0.272.

With this example of a sample of three from a population of nine, Figure 3-13 shows that there are only four outcomes for the sample proportion, namely 0/3, 1/3, 2/3 or 3/3. For those 20 possible samples with a proportion of 0, the variance of the proportion is the same as for the one sample with a proportion of 1, namely

$$v(p) = \left(\frac{9-3}{9}\right) \frac{0\,(1-0)}{3-1} = \left(\frac{9-3}{9}\right) \frac{1\,(1-1)}{3-1} = 0$$

In both instances, with a variance of 0 the *se(p)* is 0. Thus for 21 of the 84 possible samples, the standard error of the proportion in the small example is 0. For the remaining 63 possible samples the proportion was either 1/3 or 2/3, both having the same variance.

$$v(p) = \left(\frac{9-3}{9}\right) \frac{\frac{1}{3}\left(1-\frac{1}{3}\right)}{3-1} = \left(\frac{9-3}{9}\right) \frac{\frac{2}{3}\left(1-\frac{2}{3}\right)}{3-1} = 0.074$$

For these 63 possible samples, the standard error of the proportion is

$$se(p) = \sqrt{v(p)} = 0.272$$

Thus with a sample of three from a population of nine, there are only two values for the *se(p)*, namely 0 or 0.272, as shown in the upper right of Figure 3-13. Such limited values for a binomial variable do not occur when the population and samples are larger, as occurs with typical rapid surveys. Nevertheless, the example remains useful for demonstrating the calculations.

Using the standard error, the confidence intervals are calculated for each of the 84 possible samples. The confidence interval is derived using Formula 3.24.

$$CI(p) = p \pm t\ (se(p)) \tag{3.24}$$

where *p* is the proportion, *se(p)* is the standard error and *t* is a statistic corresponding to the *z*-value with the sample is exceptionally small, as noted in Figure 3-9.

To calculate 95% confidence intervals for each of the 84 possible samples, we use a *t-value* of 4.30, much larger than a *z-value* of 1.96 typically used with 95% confidence intervals. The confidence interval for the sample Joe- Hal-Roy with a proportion 0.67 is

$$CI(p) = 0.67 \pm 4.30\,(0.272)$$

or a lower limit of 0 and an upper limit of 1. Unfortunately, because the sample of three is so small, the confidence interval is extremely wide, and hence is of little use, other than to demonstrate the calculations. The 84 confidence intervals are shown in the bottom of Figure 3-13. With such a small sample, most of the confidence intervals are very wide, extending from 0 to 1. In 21 of the 84 samples the standard error was 0. For these 21 samples, the confidence interval did not bracket 0.33, the true proportion in the population. The 21 samples are identified in Figure 3-23 with an asterisk. It is clear from viewing Figure 3-13 that for this small sample of three addicts drawn from a population of nine addicts, only a 75% confidence interval could be derived. That is in 63 times out of 84 (75%), the surveyor would be confident that the created interval brackets the true value. For larger surveys with billions and trillions of possible samples, 95% of the confidence intervals that would be derived using a *z-value* of 1.96 would bracket the true value in the sampled population, assuming there was no bias in the sample selection.

3.3.3 Ratio Data

What comes next is a different way of viewing the data, introduced earlier in Chapter 2. Rather that considering the mean number of injections or the mean number of shared injections, we will determine the proportion of injections that were shared. This proportion is actually a ratio of two random variables: shared injections (variable *y*) and total injections (variable *x*). Notice that the sample is of addicts, not injections. Thus the three addicts in the sample may have different numbers of shared injections (one variable) and of total injections (a second variable). Yet the sample is only comprised of three addicts. Later in this chapter you will learn the difference between the units that are sampled, given here as addicts, and the units that are analyzed, namely total and shared injections. The former are termed *sampling units* while the latter are *elementary units*.

Figure 3-14

When considering injections rather than people, the data can be viewed as in Figure 3-14. The ratio of the two variables "shared" to "total" is calculated with Formula 3.25.

$$R = \frac{\sum_{i=1}^{N} Y_i}{\sum_{i=1}^{N} X_i} \qquad (3.25)$$

where *R* is the ratio of variable *Y* to *X* in the population, Y_i is the value of variable *Y* in person *i*, X_i is the value of variable *X* in person *i*, *N* is the number of persons in the population, and Σ is the sum symbol indicating to add the values for each person from 1 to *N*.

Figure 3-15

As shown in Figure 3-15, there were 27 shared injections during the past two week and 91 total injections, for a true ratio in the population of 0.297. Notice that the ratio is given as a proportion, but a proportion of injections, not of people.

The ratio of shared to total injections is calculated for each of the 84 possible samples using Formula 3.26.

$$r = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i} \qquad (3.26)$$

For the sample Joe- Hal- Roy, the *ratio estimator* (that is, our estimate of the ratio) is calculated as

$$r = \frac{6 + 2 + 5}{10 + 8 + 12} = \frac{13}{30} = 0.43$$

Notice that this is an estimate of the proportion of injections that is shared. The number could also be multiplied times 100 to derive the percentage of injections that are shared (i.e., 43%).

Figure 3-16

The ratio for the 84 possible samples is shown in the upper left of Figure 3-16. Here the mean of all possible sample ratios is slightly different from the ratio in the total population. While the difference is small (that is, 0.297 in the population versus 0.296 in mean of all possible samples), there is a difference nevertheless. Ratio estimators may be slightly biased, so that the average of all possible samples is not the same as the true ratio in the population. When the sample size is larger, however, the bias becomes minimal. The issue of bias in ratio estimators will be further addressed later in this chapter.

Next the variance of the ratio for each of the 84 possible samples will be calculated, followed by the standard errors and confidence intervals. The variance formula is different from those presented earlier. The formula will be presented here, but will await Chapter 5 to be discussed in more detail.

$$v(r) = \frac{N-n}{N} \frac{\sum_{i=1}^{n} (y_i - r x_i)^2}{n\ (n-1)\ \bar{x}^2} \qquad (3.27)$$

The *finite population correction* term in Formula 3.27 is *(N-n)/N*, the ratio in the sample is *r*, and the other terms are as previously defined. For the sample Joe-Hal-Roy, the variance of the sample ratio is

$$v(r) = \frac{9-3}{9} \; \frac{(6 - 0.43\, x\, 10)^2 + (2 - 0.43\, x\, 8)^2 + (5 - 0.43\, x\, 12)^2}{3\,(3-1)\, 10^2} = 0.0055$$

The standard error of the ratio, *se(r),* is the square root of the variance, *v(r)* as shown in Formula 3.28.

$$se(r) = \sqrt{v(r)} \tag{3.28}$$

For the sample Joe-Hal-Roy, the standard error is the square root of 0.0055 or 0.074. The distribution of standard errors for all possible samples is shown in the upper right of Figure 3-16.

The confidence interval for the ratio, *CI(r)*, is derived using Formula 3.29.

$$CI(r) = r \pm t\ (se(r)) \tag{3.29}$$

where *se(r)* is the standard error and *t* is the *t*-value for the stated sample size. For the sample Joe-Hal-Roy, the confidence interval is..

$$CI(r) = 0.43 \pm 4.30\, x\, 0.074$$

or a lower limit of 0.11 and an upper limit of 0.75. Confidence intervals for all 84 possible samples are shown in the lower section of Figure 3-16. The vertical axis with “Ratio (Shared/Total)” extends from a low of -0.4 to a high of 1.0, appropriate for a ratio but different from a proportion that goes from 0 to 1. While none of the point estimates of the sample ratios extend beyond 0 or 1, some of the confidence intervals do. This issue will be addressed in Figure 3-17. Also observe that 5 of the 84 confidence intervals (6%) do not bracket the true value in the population. Thus 94% of the intervals enclosed the true value. The interpretation is that in advance of sampling, we can be 94% confident that intervals derived with Formula 3.29 would bracket the true value. With larger samples from larger populations, the confidence intervals created with Formula 3.29 , but using the *z-value* of 1.96, would bracket the true value 95 percent of the time, assuming of course that there was no bias in the sample selection.

Figure 3-17

When doing rapid surveys, the analysis is often of a ratio estimator appearing as a proportion. That is, the ratio estimator seems like a variable with values between 0 and 1, even though in reality the range could be much wider. Since the point estimate for a ratio estimator that acts like a proportion lies between 0 and 1, the problem lies with the confidence limits, both on the high and low end. Since a proportion above 1 or below 0 does not make sense, the range of the confidence limits in rapid surveys is typically truncated, as shown in Figure 3-17. Note that this truncation has no effect on the confidence that the presented intervals bracket the true ratio in the population.

3.4 SAMPLES AND ELEMENTS

In the example of drug addicts, the starting point was a population of nine addicts from which three addicts were randomly selected. The process of sampling to identify the prevalence of HIV infection is shown in Figure 3-18. In order to sample from the population of nine addicts, all must first be assembled in a list termed a *frame*. From this frame, three units are sampled. Each unit is termed a *sampling unit* while the group of sampling units is termed the *sample*. In Figure 3-18, the sampling unit is a drug addict and the sample is three addicts. The units to be analyzed, termed *elementary units*, are drug addicts (or persons) and are the same as the sampling units. The group of elementary units is referred to as *elements*. In this example, the sample is the same as the elements. That is, the group selected in the sample is the same as the group to be analyzed. In Figure 3-18, one addict is HIV positive and the other two are not. Thus the population prevalence (or proportion) is 1/3 or 0.33, the numerator being HIV infected addicts and the denominator being sampled addicts. Because a sample of three addicts was selected, the size of the denominator is fixed as three. In estimating HIV prevalence, the numerator varies from one possible sample to the next. That is, the denominator is fixed but the numerator is a random variable.

Figure 3-18

A slightly different situation is shown in Figure 3-19. Here the sample is also three addicts but the elements are nine injections. If the analysis is of the mean number of injections per addict or mean number of shared injections per addict, the derivation is the mean for an equal interval variable. While the denominator remains fixed at three addicts, the numerator varies from one possible sample to the next. That is, the numerator is a random variable, but not the denominator.

Figure 3-19

The elements can also be analyzed independent of the sample. Specifically, the number of shared injections can be related to the number of total injections. This is a proportion, but using injections rather than persons. The denominator of the proportion is the number of injections which varies from one sample to the next, depending which addicts were included. In this proportion both the numerator and denominator are random variables. As a result, the data are analyzed as a *ratio estimator*, or a ratio of one random variable to another.

Figure 3-20

So what happens when the simple random sample is of *households* rather than of persons? In Figure 3-20 there is a population of households. To sample from this population, the households are arranged in a list or *frame* and households are sampled randomly from the frame. For this example, the sample consists of 30 households, selected randomly from the population of many households. For the analysis, the interest is not in characteristics of the housing unit. Instead information is wanted about the about the household occupants, namely people. Thus the analysis is of elements in the households, although only households were sampled. For such a survey, the analysis is of a *ratio estimator*, or comparison of two random variables. The random variable in the denominator is the number of persons in the sampled households and the random variable in the numerator is the number of persons with the attribute being studied.

3.5 SURVEY OF SMOKING BEHAVIOR

In Section 3.3, the population to be sampled was addicts. Yet for one analysis – shared to total injections – the data were viewed as a ratio, although the results were presented as a proportion or percentage. You observed in Figure 3-16 that a confidence interval can be created for a ratio estimator, although in our small example, the interval for some of the possible samples was very wide. You also noted in Figure 3-16 that there was a small bias in the ratio with the average value of all possible samples (i.e., 0.296) being slightly different from the true value in the population (i.e., 0.297). This section will further explore ratio estimators, using a survey of smoking behavior to illustrate the various issues and principals.

Figure 3-21

3.5.1 Sample of Persons

Before considering ratio estimators, assume that a sample of 90 persons was sampled by simple random selection from a population frame of 3,000 persons as shown in Figure 3-21. The intent is to determine the prevalence of current cigarette smoking behavior. Thus the sample units and elementary units are both persons and the outcome of the analysis is a binomial variable (i.e., smoker = 1, non-smoker = 0). In this population, 54% currently smoke. The sampling is done with replacement, discarding order. Thus using Formula 3.9, the number of possible samples of 90 from a population 3,000 is,

$$\frac{3{,}000!}{90!\ (3{,}000 - 90)!} = \frac{3{,}000\ (2{,}999)\ (2{,}998)\ ...\ (2{,}912)\ (2{,}911)\ x\ 2{,}910!}{90\ (89)\ (88)\ ...(2)\ (1)\ x\ 2{,}910!} = 1.5\ x\ 10^{174}$$

a very large number of possible samples! For each of these possible, we could derive the mean, standard error and 95% confidence interval. Yet doing so would be a formidable task, and the certainly the results could not be viewed graphically. Instead, the population is replicated in a computer spreadsheet model, and 100 surveys of 90 persons each are randomly selected from the frame of all possible samples for graphic presentation. To derive the 95% confidence interval, the finite population correction term is included and a *t-value* of 1.986 is used instead of the *z-value* of 1.96 (see Figure 3-9, sample size of 90). The 95% confidence interval for each of the 100 surveys is derived as (see Formulas 3.22 and 3.24),

$$CI(p) = p \pm 1.986 \sqrt{\left(\frac{3{,}000 - 90}{3{,}000}\right) \frac{p\,(1 - p)}{90 - 1}}$$

with the 100 confidence intervals as shown in Figure 3-22.

Figure 3-22

Observe that the average standard error for the 100 surveys is 5.2% which in later discussion will be termed “medium variability.” The mean of the 100 samples was 53.1% smokers, nearly identical and within random statistical variation of the 54% who smoked in the total population. Also notice that five of the 100 confidence intervals did not bracket the true value of 54% in the population, as would be expected when deriving 95% confidence intervals. Figure 3-22 will serve as a comparison, when next we consider a household survey with elementary units different from sampling units.

3.5.2 Sample of Households

Assume that 30 households (HH) with an average of three persons per HH were drawn in a simple random sample from a population frame of 1,000 households. As previously shown in Figure 3-20, in such a survey the sampling unit of a household is different from the elementary unit of a person. In fact, in such surveys the people in a household are represented by two variables, the first being the existence of each person and the second being the smoking status of each person. Thus the proportion who smoke is derived as a ratio of two variables within the selected households, namely smoking status and existing persons, and as such is a *ratio estimator*. When analyzing a ratio estimator, a slightly different variance formula is used (similar to Formula 3.27), to be further explained in Chapter 5.

$$v(p) = \left(\frac{N - n}{N}\right) \frac{\sum_{i=1}^{n} (a_i - p\,m_i)^2}{n(n-1)\bar{m}^2} \qquad (3.30)$$

where *N* is the number of households in the population, *n* is the number of households in the sample, a_i is the number who smoke in each household, *p* is the proportion who smoke in the total sample, m_i is the number who exist in each household, and $\bar{m}$ is the average number of persons per household.

Figure 3-23

Furthermore, the households in the population are of five occupancy sizes, as shown in Figure 3-23. Each group in the population resides in 200 homes, and in the total population of 1,000 households the average is three persons per household. Thus the simple random samples of persons and households have the same number of persons (i.e., 3,000), although in the household survey the people are distributed in 1,000 households.

Because the household survey is a sample of 30 households from a population of 1,000 households, the number of possible samples is not as great as in when doing a simple random sample of 90 persons from a population of 3,000 people. Again using Formula 3.9, the number of possible samples of 30 from a population 1,000 is,

$$\frac{1{,}000!}{30!\ (1{,}000 - 30)!} = \frac{1{,}000\ (999)\ (998)\ ...\ (972)\ (971)\ x\ 970!}{30\ (29)\ (28)\ ...(2)\ (1)\ x\ 970!} = 8.1x\ 10^{56}$$

still a huge number of possible samples. As before with the simple random sample of persons, a computer spreadsheet model is used to randomly select and present 100 of the possible samples of 30 households from the population of 1,000 households. The 95% confidence interval will again use a *t-value* instead of the *z-value* used in larger rapid surveys, but this time of 2.0452 (see Figure 3-9, sample size is 30). The 95% confidence interval for each of the 100 household surveys is derived as (see Formulas 3.22 and 3.24),

$$CI(p) = p \pm 2.0452 \sqrt{\left(\frac{1{,}000 - 30}{1{,}000}\right) \frac{p\,(1-p)}{30-1}}$$

Variability of Smoking. Measures such as the variance, standard error and confidence interval provide a numeric assessment of the variability of smoking behavior in the population. In a simple random sample of persons, the variability is observed when comparing one individual to another. In contrast, the variability of persons in a simple random sample of households is expressed both within households by the smoking patterns of the people that occupy the household, and between households, comparing the proportion of residents who smoke in one households versus the others.

Figure 3-24

In Figure 3-22, the variability of the persons selected in a simple random sample was presented as 100 different 95% confidence intervals and summarized with the average standard error value of 5.2%. This level of variation was termed "medium variability." Figure 3-24 shows two situations where the distribution of smoking behavior is the same in the population of 3,000 persons (the top section) and in the 1,000 households (the bottom section). The only change in the bottom section is that households were placed around members of the population. Thus the people within households are as variable in their smoking patterns as in the population. Or restated, the chance of being a smoker within a household is independent of the smoking status of other members of the household. This may not seem logical given human behavior, but instead is used to illustrate a notion. Since the level of variability in the population of 3,000 persons and 1,000 households with an average of three persons per household is the same, the household surveys should also exhibit "medium variability."

Figure 3-25

The distribution of smokers within each group of 200 households is shown in Figure 3-25. The figure is divided into four sections, each describing the population household distributions that is associated with different levels of smoking variability in random sample surveys. The first distribution to be considered is of medium variability in part B.

Medium Variability. In this example, 54% of the population smokes. Assuming that people in small households are just as likely to smoke as people in large households, then the percent smokers in each of the five-sized households should also be 54%. Figure 3-25 B shows the distribution of smokers in households by household size, for household surveys in which the variability is the same as in a simple random sample of persons (i.e., medium variability).

Figure 3-26

When drawing 100 samples of 30 households from the billions of possible samples, the values of the 95% confidence intervals are presented in Figure 3-26 B. The dark horizontal line shows the true value in the population, or 54% smokers. Notice that the standard error of 5.1% is nearly identical to the standard error of 5.2% for the simple random sample (see Figure 3-22). The minor difference is due to random variation in the sampling process. The mean of the 100 samples is 53.9%, very close to the true value of 54%. Finally, notice that six of the 100 confidence intervals in Figure 2-26 B do not bracket the true value of 54% smokers from this population, nearly the same as the five that would be expected by chance. If many more sets of 100 surveys would have been drawn from all possible samples, the percentage of 95% confidence intervals that do not bracket the true value would likely be in the 4-6 percent range, averaging 5 percent.

So far we have considered the situation where the level of variability in a household survey (termed "medium" in magnitude) is identical to a random sample of persons in the population. What if this is not true? What if the variability in a household survey is greater or less than the variability of a survey of persons? To understand the variability of household surveys, we need next to consider variability both within households and between households.

3.5.1 Variability Within and Between Households

The variability of a binomial variable such as smoking is derived using variance formula for a proportion (see Formula 3.22). Leaving out the finite population correction term, the v(p) formula is reduced to

$$v(p) = \frac{p\,(1-p)}{n-1} = \frac{p\,q}{n-1}$$

This formula shows that for any given household size, the variance of the proportion who smoke is depended on the size of the numerator, namely p x q, distributed as shown in Figure 3-27 for five-person households.

Figure 3-27

The variance within a five-person household will be zero if either none of the occupants are smokers or if all of the occupants are smokers. Conversely, the variance within households will be largest if either two or three of the five household members are smokers.

Figure 3-28

There are 200 five-person households in the total population. The variability within and between these 200 households is shown in Figure 3-28 for the population with 54% smokers. If most of the 200 households have high variability within households (i.e., namely have 2-3 smokers), then the 200 households will be distributed by smoking status as in Figure 3-28, bottom left. That is, most of the households will have 3 smokers, many will have 2 smokers, and a handful will have 4 smokers. The variability of smokers in these 200 five-person homes is less (i.e., low variability) than the variability of the five-person homes in the previous example (i.e., medium variability), with variability the same as in the general population, as seen in Sections A and B of Figure 3-25. The same principal of higher variability within households resulting in lower variability between households is also seen in Figure 3-5 A and B, comparing the one-, two-, three- and four-person households in the two sections.

The right section of Figure 3-28 presents a different situation. When five-person households have either no smokers or all smokers, the variance within households is zero (i.e., low variability). Yet, when all of the 200 households are split with about half having no smokers and approximately half having five smokers, the variance between households becomes very large (i.e., high variability). Section C of Figure 3-25 shows the distribution of the five sets of 200 households when the variability is large.

These two graphs, Figures 3-25 and 3-28, illustrate a concept that is very important when planning rapid surveys. If the variable that is being considered (in our example it is “smoking”) is distributed within households in a near random manner, then the variability of a household survey will be nearly the same as the variability of a simple random sample of the population. If the variable being studied is a highly infectious disease, then households might have either all persons infected or no persons infected. In such situations, the variability would be low within households but high between households. Finally, if the variable occurs uniformly in all households with half of each household having the attribute and the other half not, then the variability will be high within the households but low between the households.

The effect of the variability between households on the 95% confidence interval derived from the household survey is shown in Figure 3-26. Section B provides the comparison group (“medium variability), or the same pattern for a household survey of 30 (with an average of three occupants per household) as a simple random sample of 90 persons. When the variability is low within households (i.e., either none or all having the attribute), Section C shows the 95% confidence intervals became very wide, exhibiting high variability. Conversely, when about half the people within a household uniformly have an attribute and the variability between households is low, Section A shows very narrow 95% confidence intervals.

The proportion who smoke in Sections A-C of Figures 3-25 and 3-26 have three things in common. First, the proportion is derived as a *ratio estimator* (i.e., ratio of two random variables). Second, the value of 54% who smoke in the underlying population of 1,000 households is the same in each of five-sized groups of 200 households. That is, in Sections A-C of the two figures, the percentage who smoking is not associated with household size, but rather is uniform by household size. Third, the 95% confidence intervals act as would be expected. That is, about 5 percent of the 100 intervals do not bracket the true value in the population. In Section A of Figure 3-26, only four 95% confidence intervals do not bracket 54%, while in Section B, six 95% confidence intervals do not bracket the true value. Finally in Section C, exactly five intervals do not bracket the true value. The difference between four, six and five non-bracketing intervals is due to chance variation in the sample of 100 surveys from the huge number of possible surveys, not due to an error in the statistical calculations.

But what happens if there is high variability (i.e., low variability within households and high variability between households) and the percentage who smoke is associated with household size? The answer to this question is addressed in Section D of Figures 3-25 and 3-26. Here the variability between households is as extreme as in Section C. Yet while the percentage who smoke in the total population remains at 54%, the percentage who smoke in the various sized households ranges between 10% in one-person households and 80% in five-person households. This change adds further distortion to the calculation of the ratio estimator, but not enough to change the usefulness of the 95% confidence intervals. As seen in Section D of Figure 3-26, the 95% confidence intervals remain wide (as would be expected with high variability), but only five out of 100 of the intervals do not bracket 54%, the true value in the population. Thus the confidence intervals still behave as expected.

Table 3-1

The analyses of the four sets of household surveys (i.e., 30 households with an average of 3 persons per household) presented in Sections A-D of Figures 3-25 and 3-26 are summarized in Table 3-1, and compared to a simple random sample of 90 persons. The average standard error for the attribute "smoking" when compared to the surveys of 90 persons is lower in household surveys with lower variability between households, the same in household surveys with medium between-household variability, and much greater in household surveys with high variability. Increasing the between household variabiliy did not result in bias when the percentage who smoked was not related to household size, since the mean percent smokers was nearly the same as the 54% in the total population. When the situation was altered so that the percentage who smoked was associated with household size, the ratio estimator of the proportion who smoked became biased, but only slightly (i.e., 52.8% versus 54%). While these observations are being made on a household survey drawn by simple random sample households, the same issues will later be addressed when considering the variability in clusters, as done in two-stage cluster surveys.

3.6 SUMMARY

In this chapter you learned about simple random sampling and what happens when the sampling unit is different from the elementary unit. Two examples were featured: a sample of three drug addicts from a population of nine addicts, and a sample of 30 households from a population of 1,000 households. You should understand how a simple random sample is selected, how the standard error and confidence interval are calculated, and the interpretation of the confidence interval. The latter is especially important for explaining your findings to others who may not have much understanding of statistics. In addition, you have learned that certain variables must be analyzed differently in a survey because they are not collected as sampling units. Instead they are elementary units that are associated with, but different from, sampling units. When the elementary unit is the unit of interest, we analyze the findings as a ratio estimator. This point will be further discussed in Chapter 5. Finally, you learned that variability of an attribute within aggregated units such as households effects variability between households, and that this variability is reflected in the size of the survey confidence interval.

Coming in Chapter 4 is a presentation of *equal probability of selection methods* and the need to select samples for rapid surveys with probability proportionate to size.

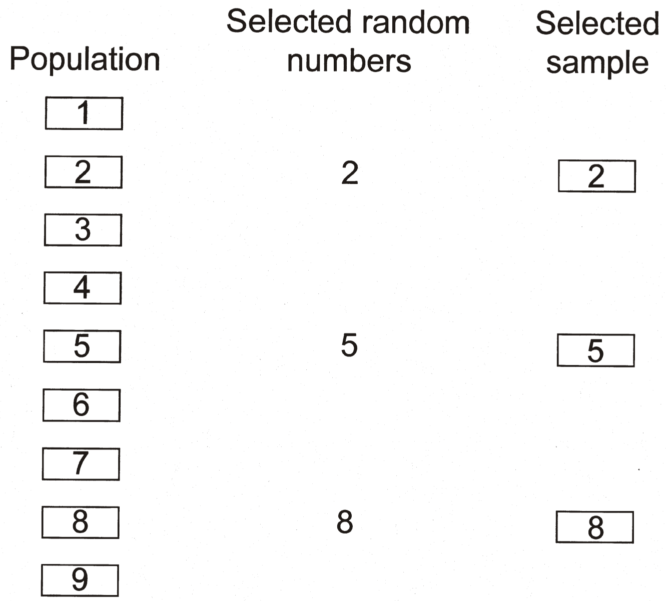
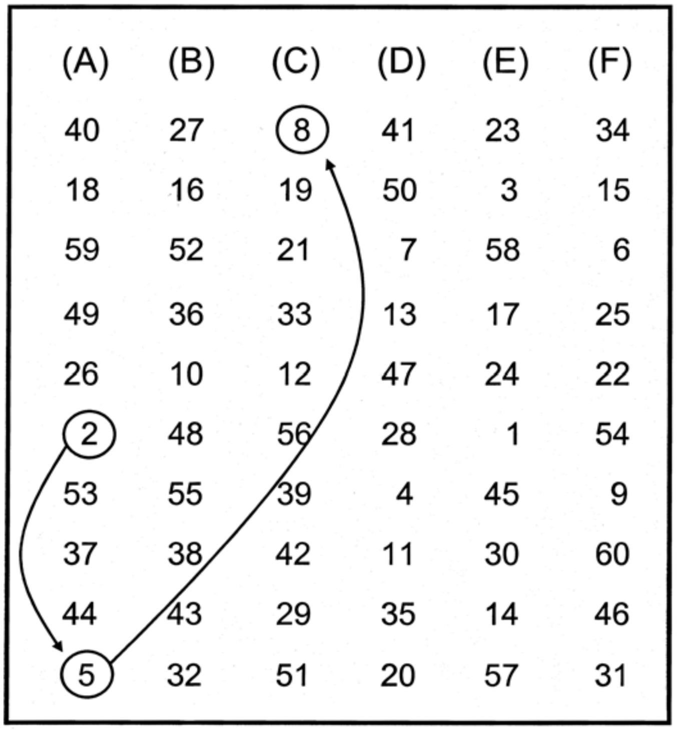


Figure 3-1. Random sample of three units from a population of nine units.



(A)	(B)	(C)	(D)	(E)	(F)
40	27	8	41	23	34
18	16	19	50	3	15
59	52	21	7	58	6
49	36	33	13	17	25
26	10	12	47	24	22
2	48	56	28	1	54
53	55	39	4	45	9
37	38	42	11	30	60
44	43	29	35	14	46
5	32	51	20	57	31

Figure 3-2. Table of random numbers.

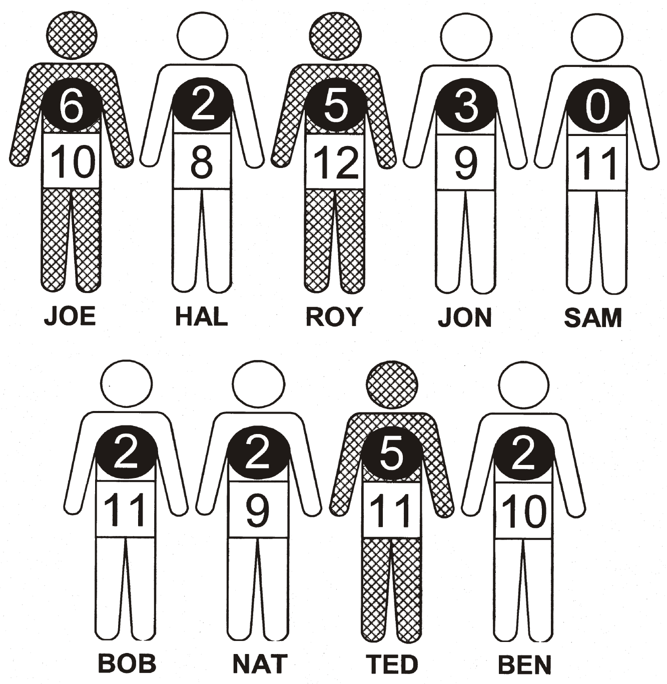


Figure 3-3. Population of nine intravenous drug addicts.

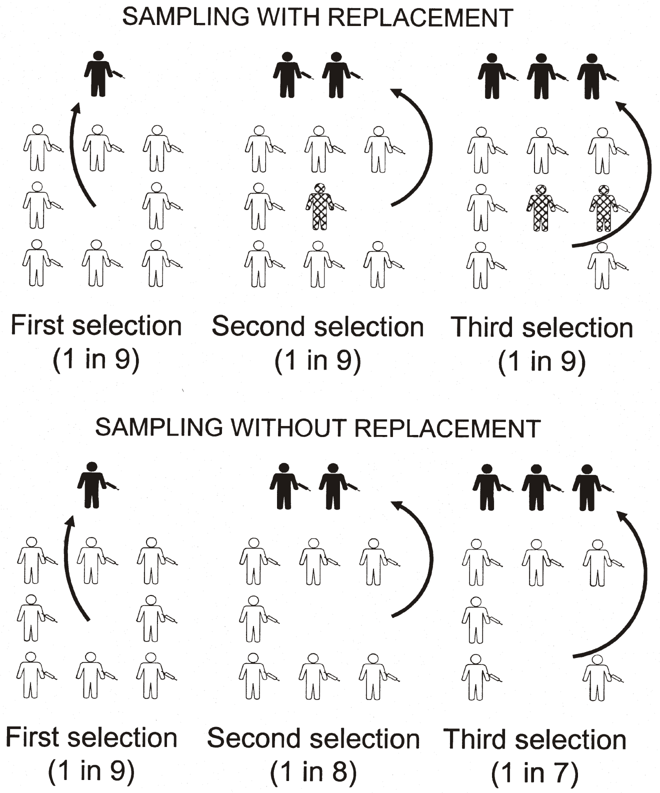


Figure 3-4. Sample of three addicts from a population of nine addicts.

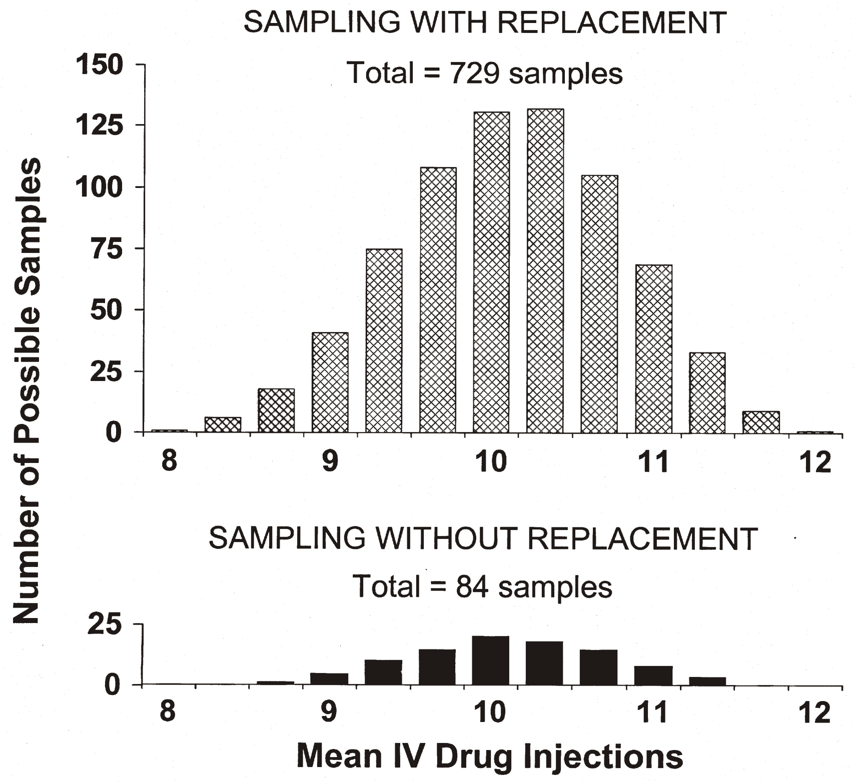


Figure 3-5. Distribution of all possible sample means *with* and *without replacement* (actual scale).

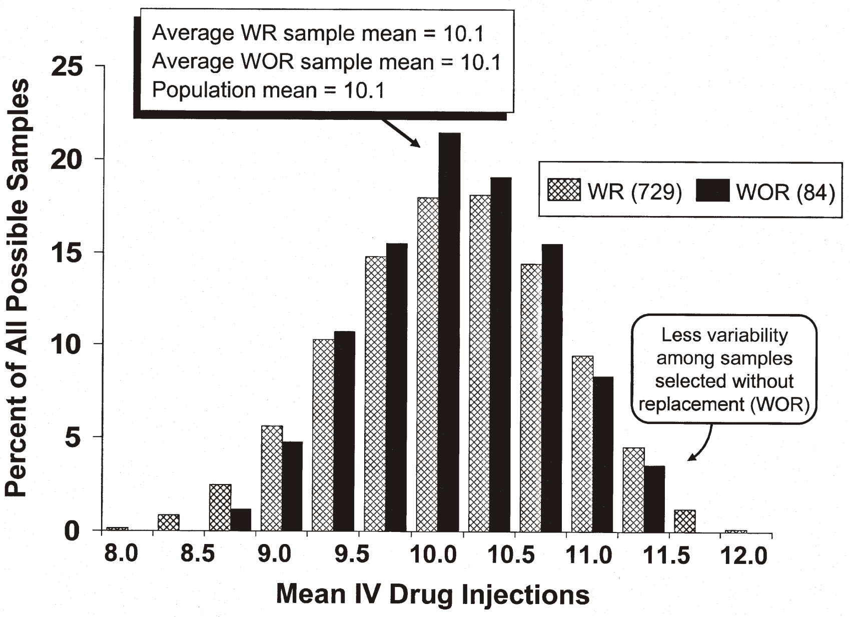


Figure 3-6. Distribution of all possible sample means *with* and *without replacement* (percentage scale).

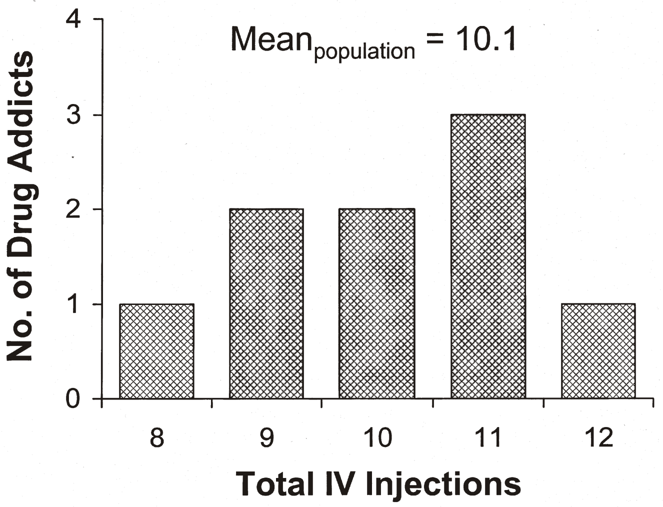


Figure 3-7. Distribution of total intravenous injections in a population of nine drug addicts.

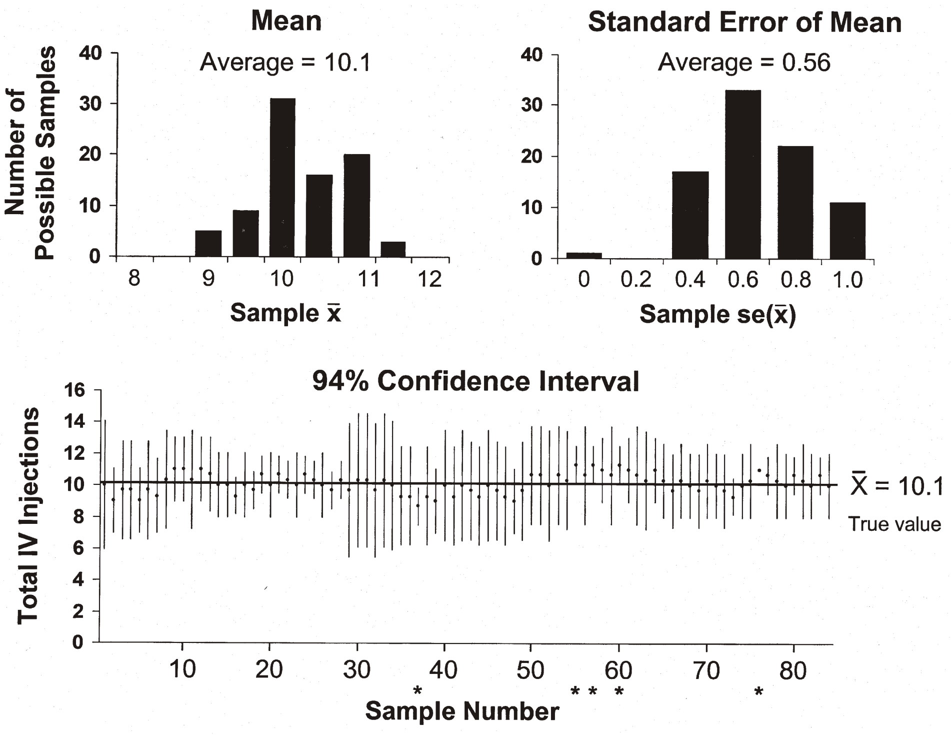


Figure 3-8. Mean, standard error, and confidence interval for mean of total injections in all possible samples of three addicts from a population of nine addicts. (* does not bracket true value)

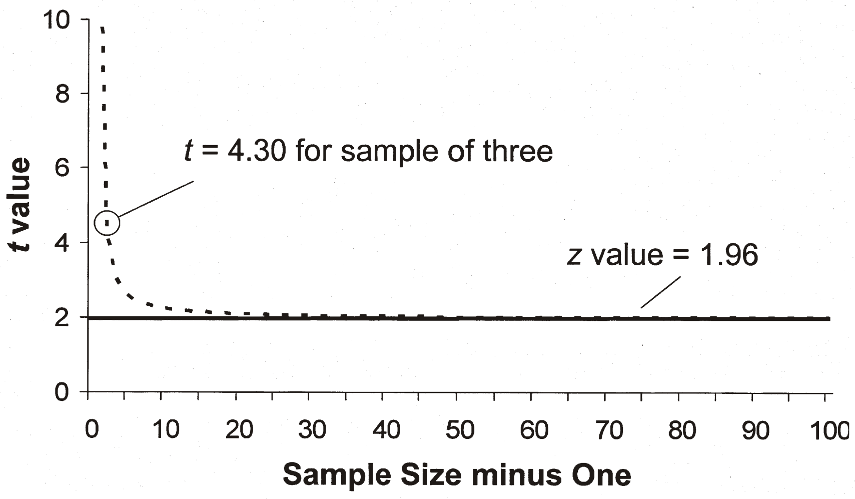


Figure 3-9. Substitution of *t-value* for *z- value* in deriving confidence interval for sample of three addicts.

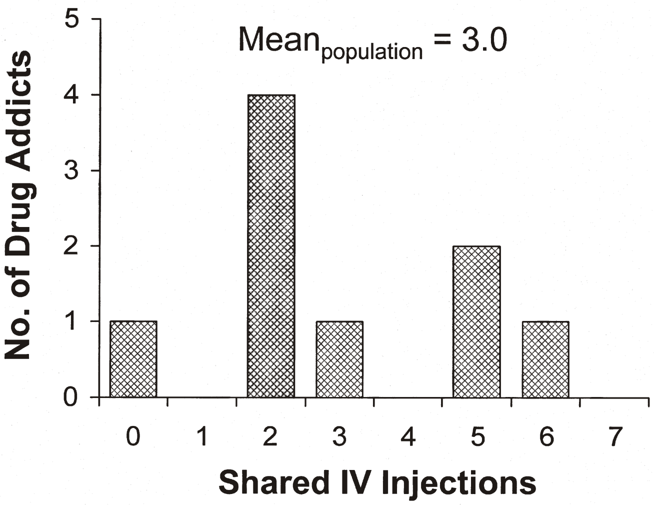


Figure 3-10. Distribution of shared intravenous injections in a population of nine drug addicts.

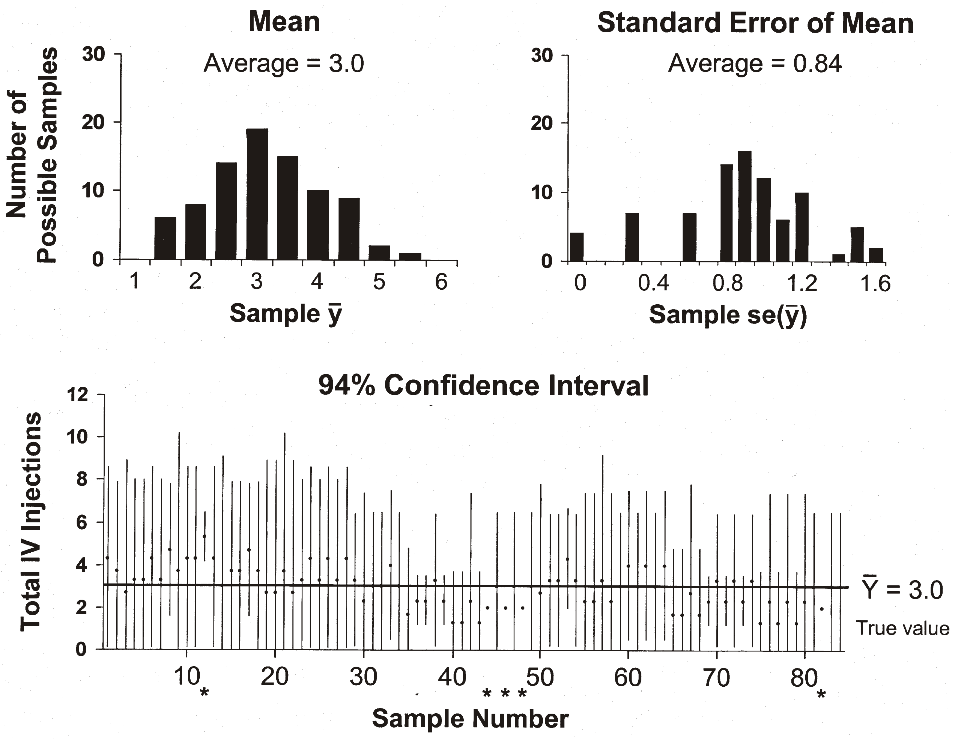


Figure 3-11. Mean, standard error, and confidence interval for mean of shared injections in all possible samples of three addicts from a population of nine addicts. (* does not bracket true value)

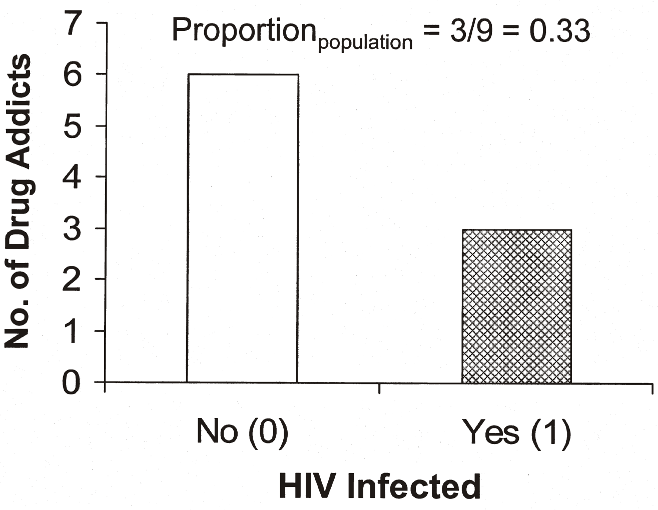


Figure 3-12. Distribution of HIV status in a population of nine drug addicts.

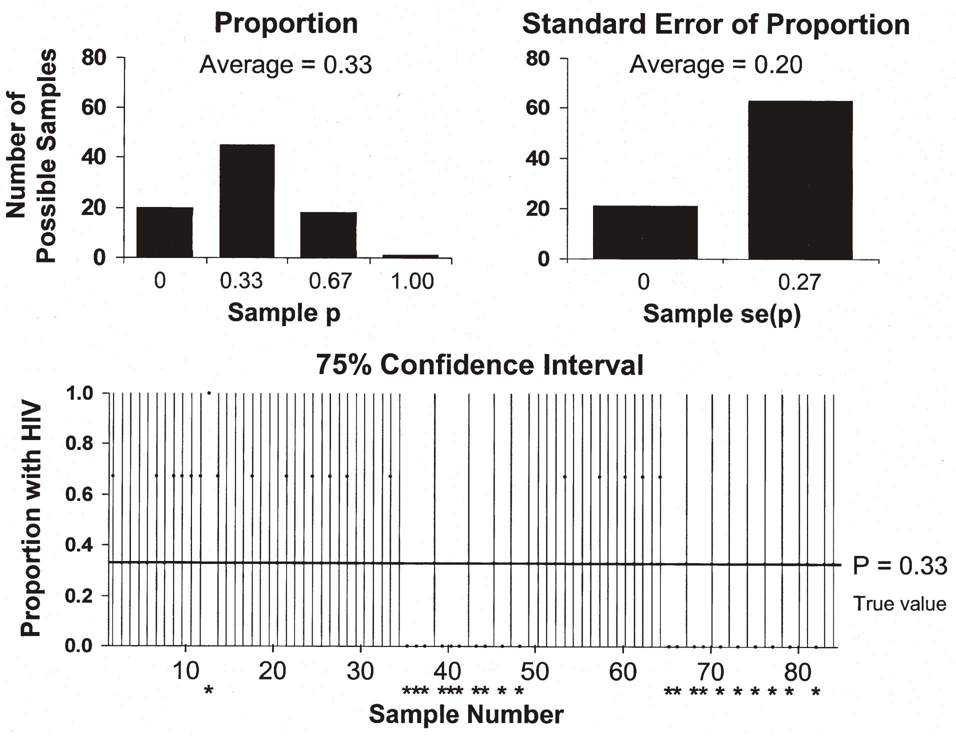


Figure 3-13. Proportion, standard error, and confidence interval for proportion with HIV in all possible samples of three addicts from a population of nine addicts. (* does not bracket true value)

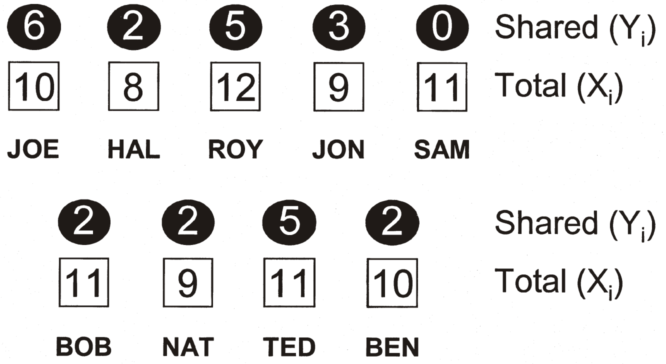


Figure 3-14. Shared and total intravenous drug injections in a population of nine drug addicts.

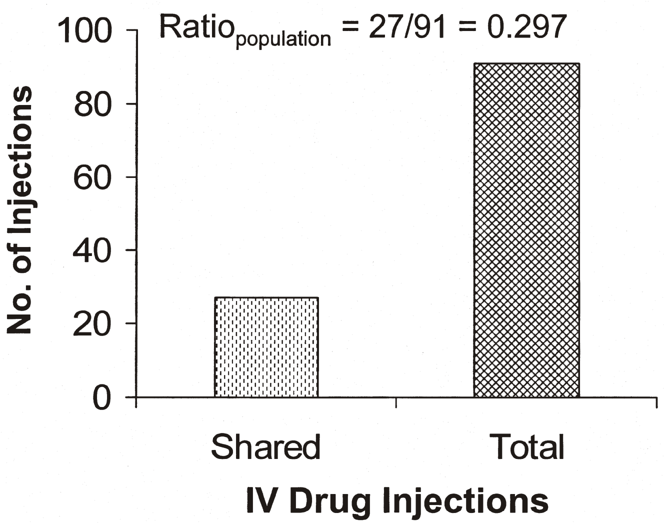


Figure 3-15. Distribution of shared and total intravenous drug injections in a population of nine drug addicts.

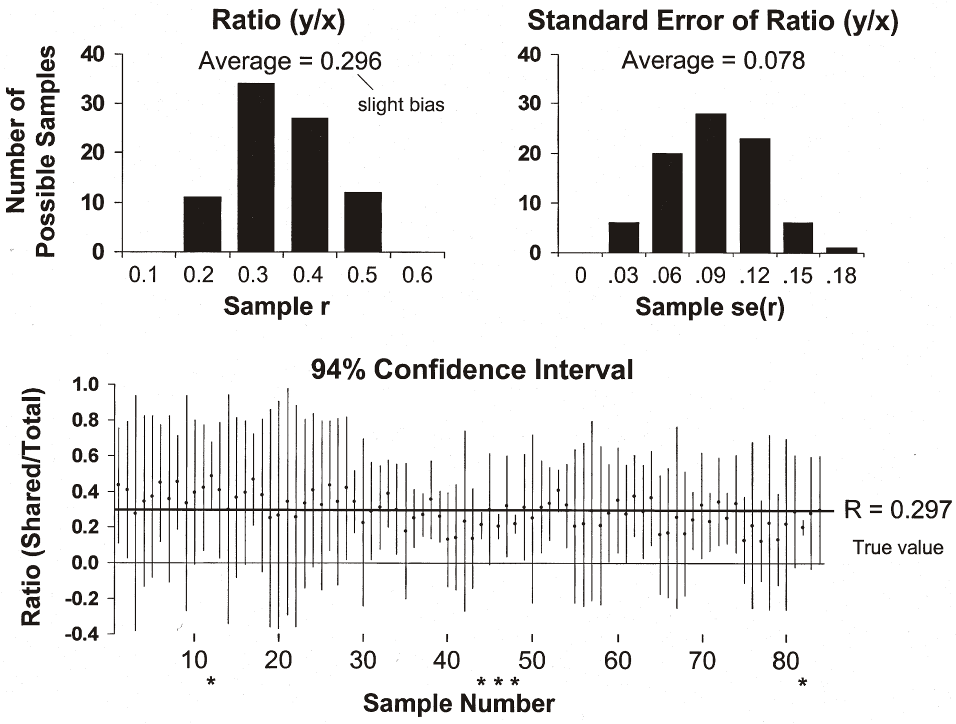


Figure 3-16. Ratio, standard error, and confidence interval for ratio of shared injections to total injections in all possible samples of three addicts from a population of nine addicts. (* does not bracket true value)

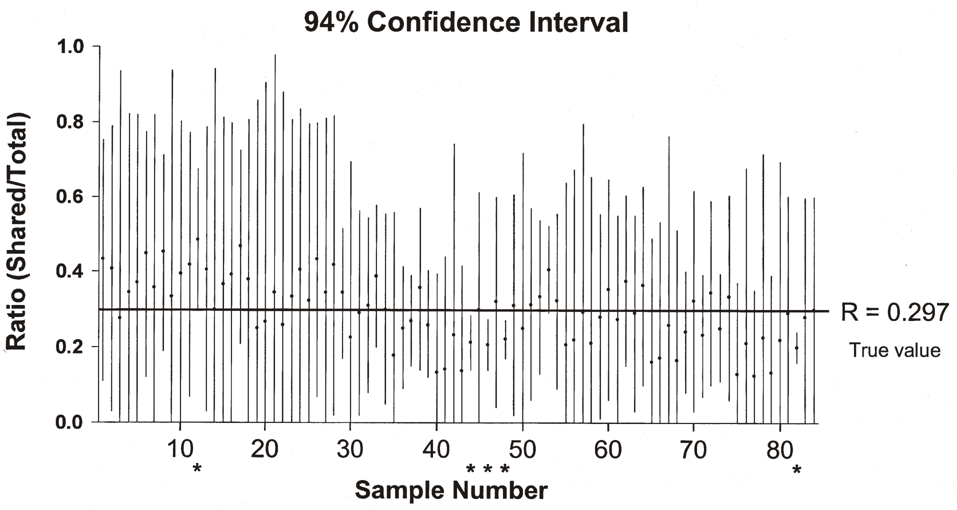


Figure 3-17. Confidence interval with range from 0 to 1 for ratio of shared injections to total injections in all possible samples of three addicts from a population of nine addicts. (*does not bracket true value)

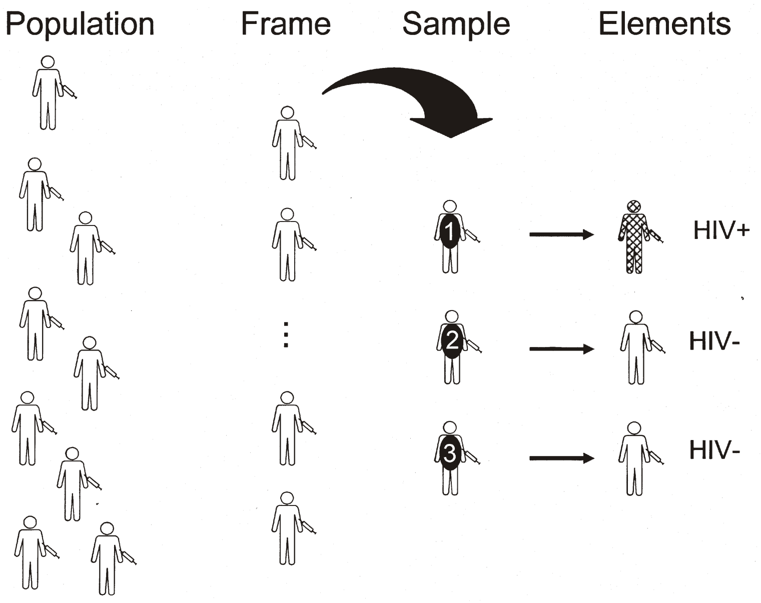


Figure 3-18. Sampling from population with sample (drug addicts) same as elements (drug addicts).

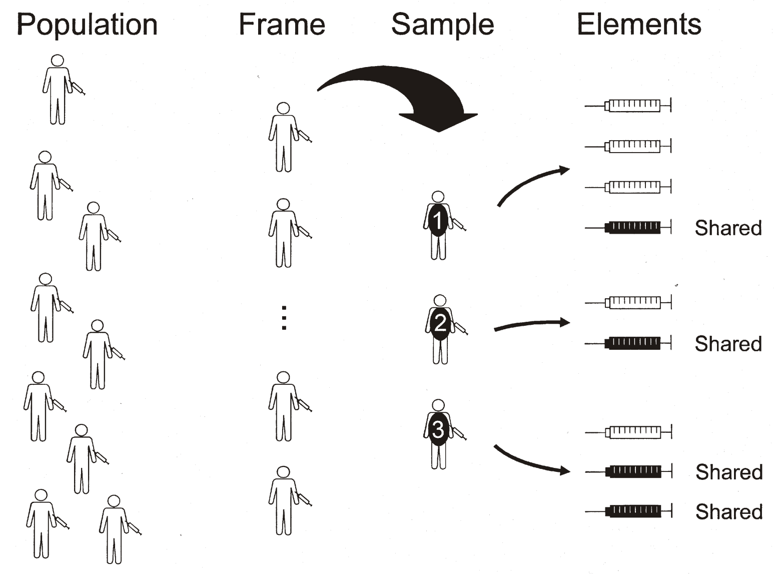


Figure 3-19. Sampling from population with sample (drug addicts) different from elements (injections).

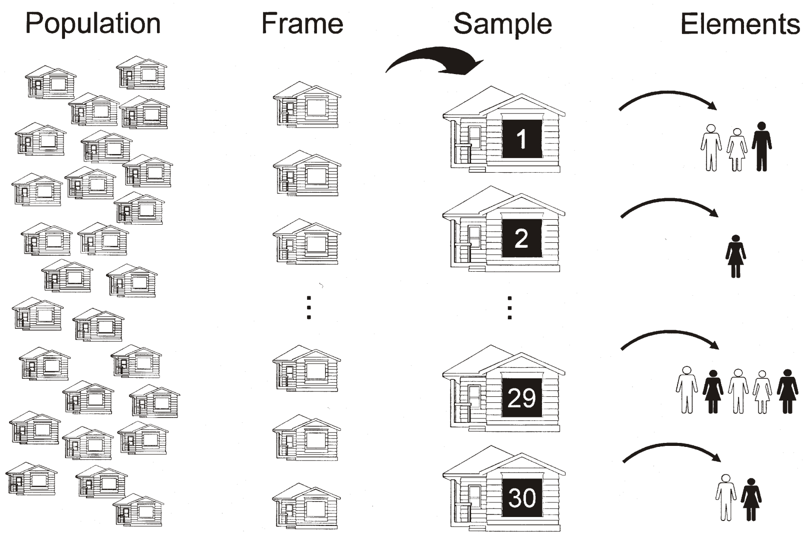


Figure 3-20. Sampling from population with sample (households) different from elements (people).

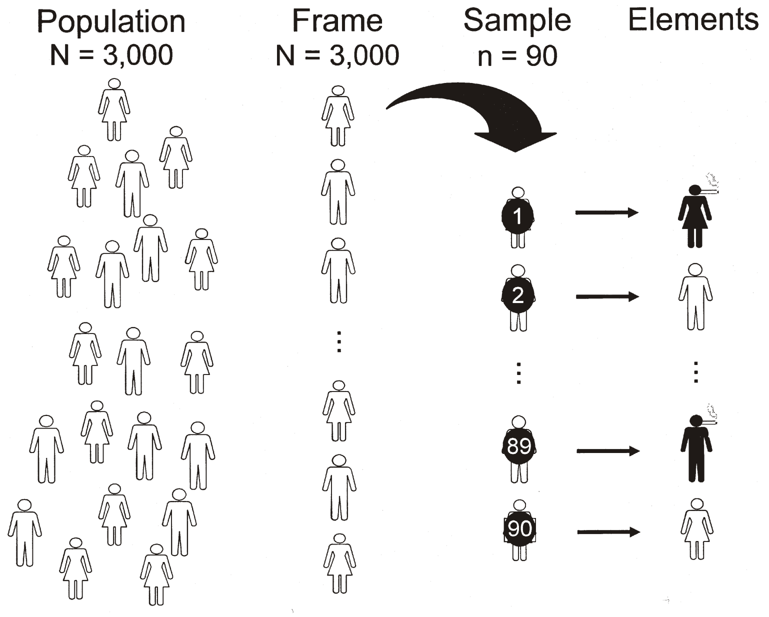


Figure 3-21. Simple random sample of 90 persons from a population of 3,000 persons, analyzed for current smoking behavior.

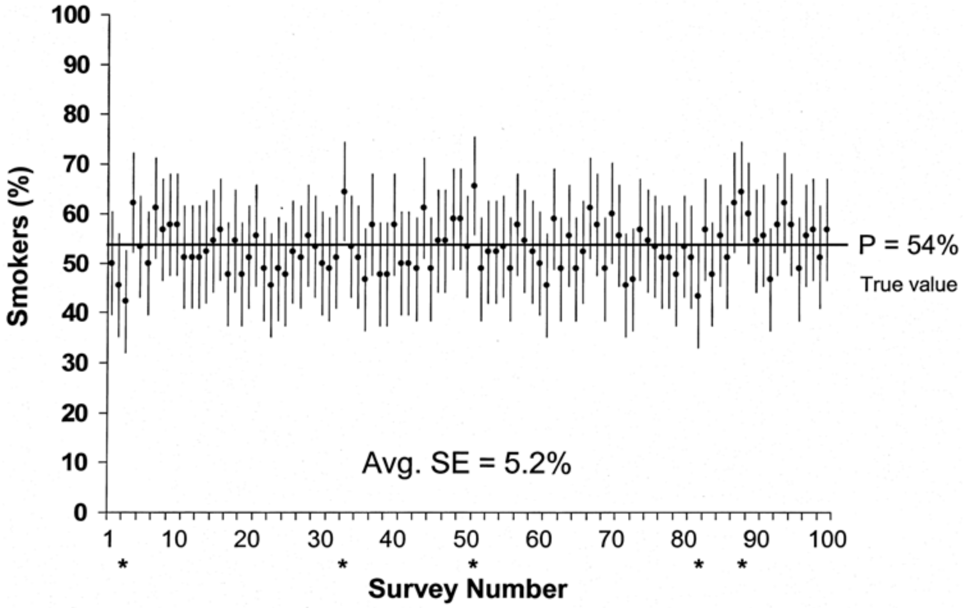


Figure 3-22. 95% confidence intervals for 100 random samples of 90 persons from a population of 3,000 in which 54% are smokers. (* does not bracket true value)

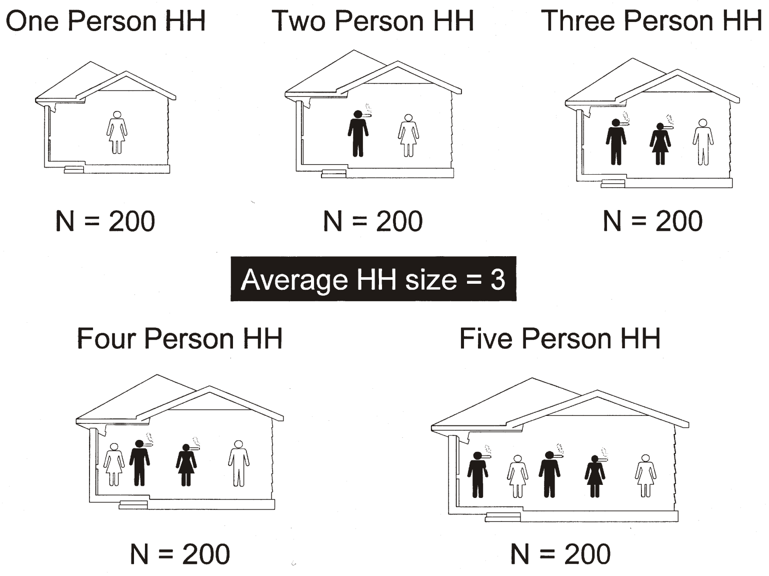
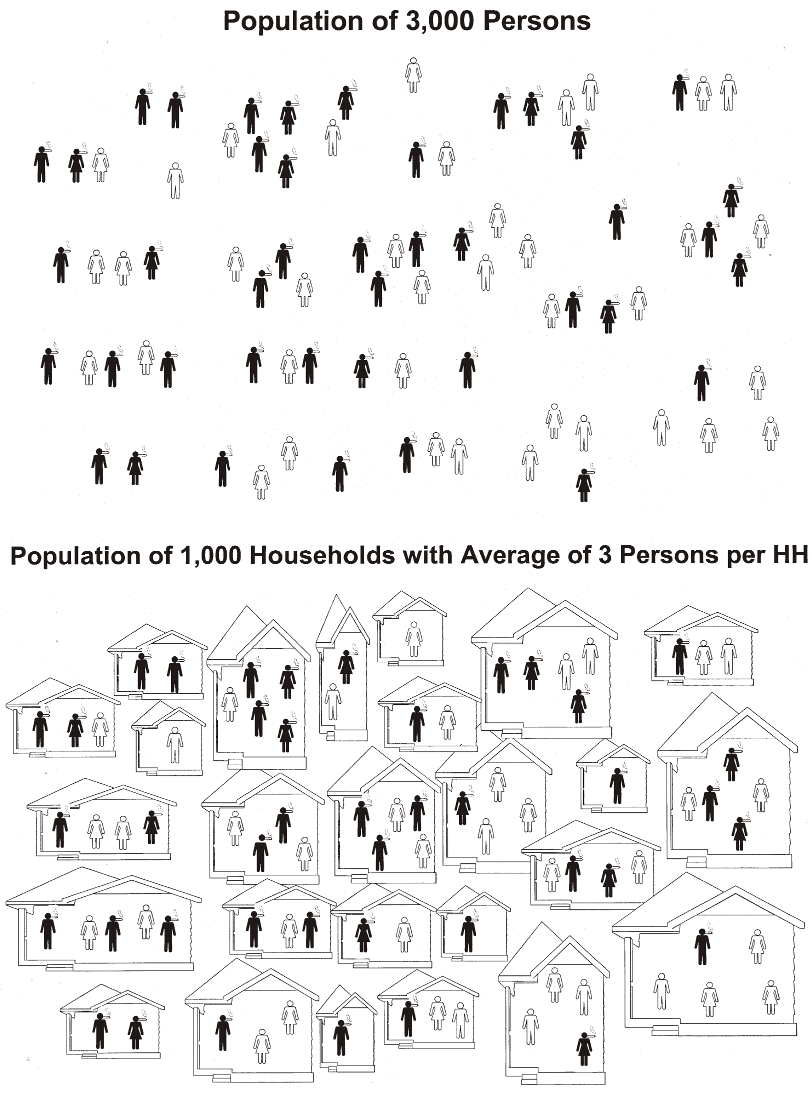


Figure 3-23. Household survey of smoking behavior in population with 1,000 households of five resident number, each group occupying 200 households.



Population of 3,000 Persons

Population of 1,000 Households with Average of 3 Persons per HH

Figure 3-24. Medium variability of smoking behavior is identical in the sampling frames of 3,000 persons (top) and 1,000 households occupied by an average of 3 persons (bottom).

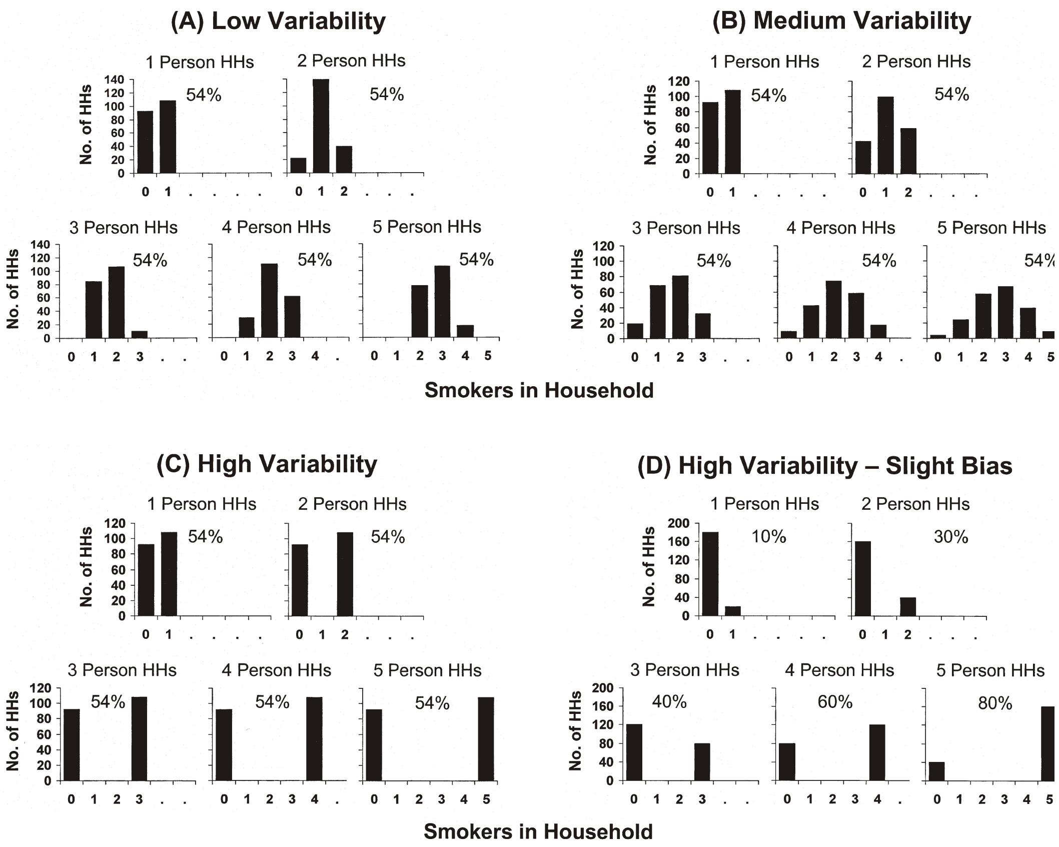


Figure 3-25. Distribution and variability of smokers in 1,000 households (HH) by HH size (each has 200 HHs) in a population in which 54% are smokers. A-D described in text.

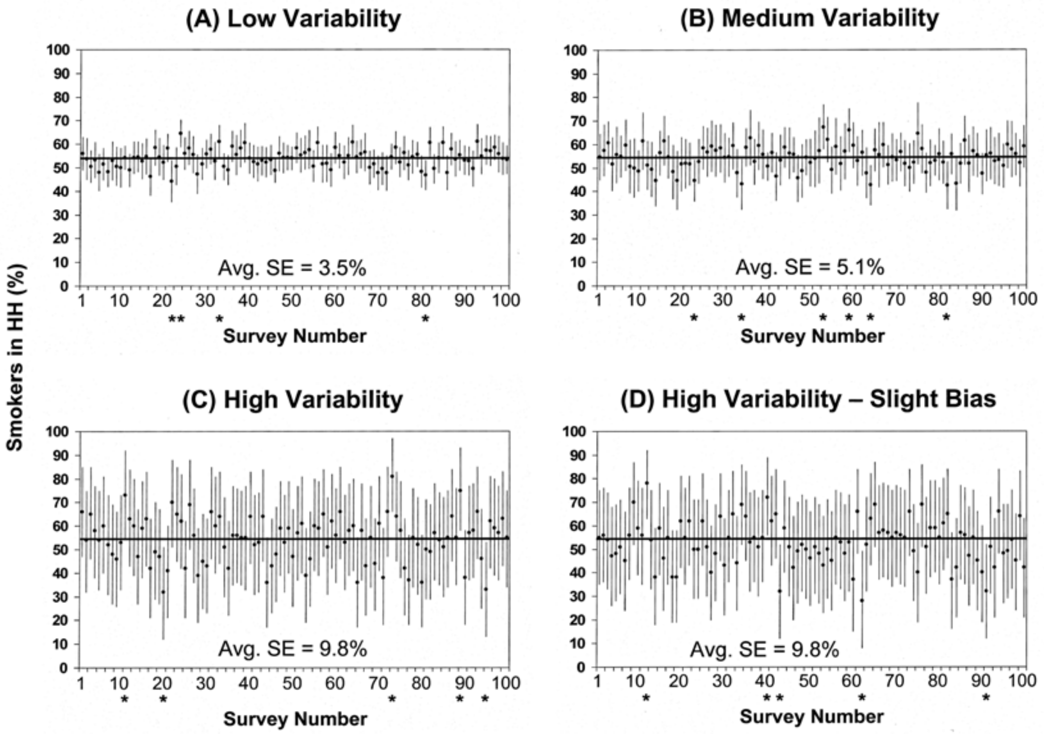


Figure 3-26. 95% confidence intervals for smoking status in 100 surveys of 30 households by distribution and variability of smokers in population of 1000 households. A-D described in text. (* does not bracket true value)

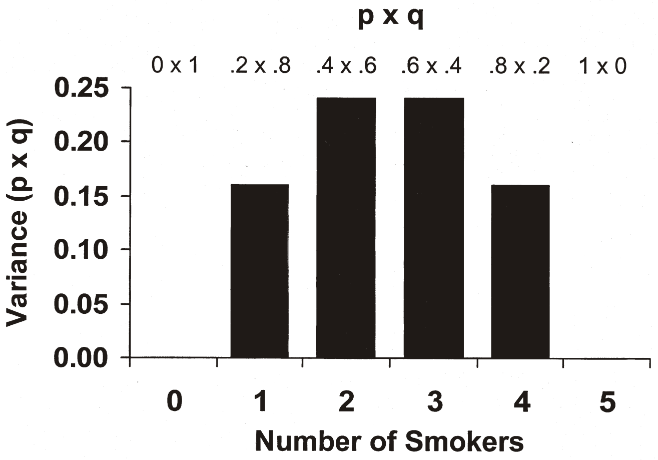


Figure 3-27. Variance of binomial variable “smoking” by number of smokers within five-person households.